

On oriented cliques with respect to push operation

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Abstract

To push a vertex v of a directed graph \vec{G} is to change the orientations of all the arcs incident with v . An oriented graph is a directed graph without any cycle of length at most 2. An oriented clique is an oriented graph whose non-adjacent vertices are connected by a directed 2-path. A push clique is an oriented clique that remains an oriented clique even if one pushes any set of vertices of it. We show that it is NP-complete to decide if an undirected graph is underlying graph of a push clique or not. We also prove that a planar push clique can have at most 8 vertices. We also provide an exhaustive list of minimal (with respect to spanning subgraph inclusion) planar push cliques.

Keyword: oriented graphs, oriented cliques, push operation, planar graphs.

1 Introduction

An *oriented graph* \vec{G} is a directed graph with no cycle of length 1 or 2. The underlying undirected simple graph of \vec{G} is denoted by G while \vec{G} is an *orientation* of G . An *oriented k -coloring* of an oriented graph \vec{G} is a mapping f from the vertex set $V(\vec{G})$ to a set of k colors such that $f(u) \neq f(v)$ whenever u and v are adjacent and, if \vec{uv} and \vec{wx} are two arcs in \vec{G} , then $f(u) = f(x)$ implies $f(v) \neq f(w)$. The *oriented chromatic number* $\chi_o(\vec{G})$ of an oriented graph \vec{G} is the smallest integer k for which \vec{G} admits an oriented k -coloring. Oriented coloring is a well studied topic (see the latest survey [10] for details).

To *push* a vertex v of a directed graph \vec{G} is to change the orientations of all the arcs (that is, to replace an arc \vec{xy} by \vec{yx}) incident with v . Push operation on directed graphs is a well studied topic [2, 5, 4].

Two orientations \vec{G} and \vec{G}' of G are in a push relation if one can obtain \vec{G}' by pushing a set of vertices of \vec{G} . The *pushable chromatic number* $\chi_p(\vec{G})$, introduced by Klostermeyer and MacGillivray [7], of an oriented graph \vec{G} is the minimum oriented chromatic number taken over all oriented graphs that are in push relation with \vec{G} .

An oriented clique, introduced by Klostermeyer and MacGillivray [6], is an oriented graph \vec{C} with $\chi_o(\vec{C}) = |\vec{C}|$. In fact, an oriented clique is characterized as those oriented graphs in which each pair of non-adjacent vertices are connected by a directed 2-path. Due to the above characterization oriented cliques can be viewed as natural objects. Moreover, they play a

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significant role in studying oriented coloring as pointed out in [8]. An undirected simple graph is called an *underlying oriented clique* if it is underlying graph of an oriented clique.

We are interested in those oriented cliques that remain invariant under the push operation, that is, those oriented cliques for which every oriented graph with a push relation with them are also oriented cliques. In fact it is easy to observe that if \vec{C} is such a clique, then $\chi_p(\vec{C}) = |\vec{C}|$. Thus, an oriented graph \vec{C} is a *push clique* if $\chi_p(\vec{C}) = |\vec{C}|$. Also an undirected simple graph is an *underlying push clique* if it is underlying graph of a push clique.

Given an undirected simple graph it is NP-hard to determine if it is an underlying oriented clique [1]. We prove an analogous result for underlying push cliques.

Theorem 1.1. *Let G be an undirected simple graph. It is NP-complete to decide if G is an underlying push clique.*

Oriented cliques of planar and outerplanar graphs are studied in details [8]. It is proved that a planar oriented clique can have at most 15 vertices [8] which positively settled a conjecture by Klostermeyer and MacGillivray [6]. Note that there exists a planar oriented clique with 15 vertices which implies that the above mentioned bound is tight. Here we prove that a push clique can have at most 8 vertices and that this bound is tight.

Theorem 1.2. *A planar push clique can have at most eight vertices. Moreover, there exists a planar push clique with eight vertices.*

Klostermeyer and MacGillivray showed that an outerplanar oriented clique can have at most seven vertices and any outerplanar oriented clique must have a particular oriented clique as a spanning subgraph [6]. Later this result was extended by providing an explicit list of eleven outerplanar graphs and proving that any outerplanar underlying oriented clique must have one of the eleven outerplanar graphs as its spanning subgraph [8]. In the same article the following question was asked: “Characterize the set L of graphs such that any planar graph is an underlying oriented clique if and only if it contains one of the graphs from L as a spanning subgraph.” Here we answer an analogous version of the above question for planar underlying push cliques.

Theorem 1.3. *An undirected planar graph G is an underlying push clique if and only if it contains one of the eighteen planar graphs depicted in Figure 1 as a spanning subgraph.*

In Section 2 we introduce some basic definitions and notations. The proofs of Theorem 1.1, 1.2 and 1.3 are given in Section 3, 4 and 5, respectively. Theorem 1.2 was published in EuroComb 2013 [9].

2 Preliminaries

For an oriented graph G every parameter we introduce below is denoted using G as a subscript. In order to simplify notation, this subscript will be dropped whenever there is no chance of confusion.

The set of all adjacent vertices of a vertex v in an oriented graph G is called its set of *neighbors* and is denoted by $N_G(v)$. If there is an arc uv , then u is an *in-neighbor* of v and v is an *out-neighbor* of u . The set of all in-neighbors and the set of all out-neighbors of v are denoted by $N_G^-(v)$ and $N_G^+(v)$, respectively. The *degree* of a vertex v in an oriented graph G , denoted by $d_G(v)$, is the number of neighbors of v in G . Naturally, the *in-degree* (resp. *out-degree*) of a vertex v in an oriented graph G , denoted by $d_G^-(v)$ (resp. $d_G^+(v)$), is the number of in-neighbors

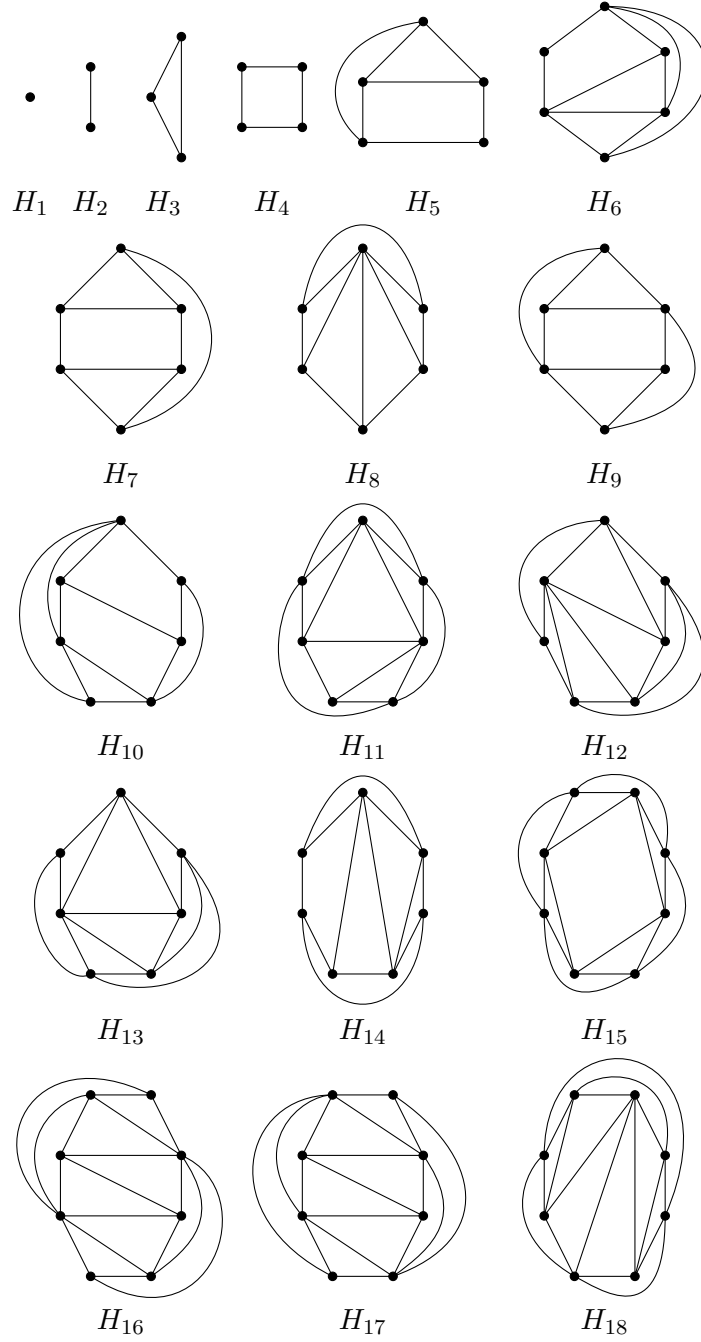


Figure 1: List of minimal planar underlying push cliques upto spanning subgraph inclusion.

(resp. out-neighbors) of v in G . The *order* $|G|$ of an oriented graph G is the cardinality of its set of vertices $V(G)$.

Two vertices u and v of an oriented graph *agree* on a third vertex w of that graph if $w \in N^\alpha(u) \cap N^\alpha(v)$ for some $\alpha \in \{+, -\}$. Two vertices u and v of an oriented graph *disagree* on a third vertex w of that graph if $w \in N^\alpha(u) \cap N^\beta(v)$ for some $\{\alpha, \beta\} = \{+, -\}$.

Now, take an oriented cycle of length 4 with arcs ab, bc, cd, ad . Note that all the oriented graphs which are in push relation with it are isomorphic to it. We call this a *special 4-cycle*. Notice that the non-adjacent vertices of a special 4-cycle always get different colors as they are always connected with a 2-dipath, no matter which vertex of the graph you push. This is in fact a necessary and sufficient condition for two non-adjacent vertices to receive two distinct colors under any oriented coloring with respect to any push relation. Thus push cliques can be characterized as those oriented graphs whose any two distinct vertices are either adjacent or part of a special 4-cycle.

3 Proof of Theorem 1.1

Given an oriented graph one can check if it is a push clique or not in polynomial time using the above mentioned characterization of a push clique.

Let G be a graph. Define G_* to be the graph obtained by adding a vertex v_* to G such that v_* is adjacent to each vertex of G . Then the following holds.

Lemma 3.1. *The graph G_* is an underlying push clique if and only if G is an underlying oriented clique.*

Proof. Assume that G_* is an underlying push clique. Let \vec{G}_* be an orientation of G_* such that \vec{G}_* is a push clique. Let \vec{G}_*' be the orientation of G_* obtained by pushing all the in-neighbors of v_* . Therefore, in \vec{G}_*' the vertices of G are all out-neighbors of v_* . As \vec{G}_*' is a push clique, each pair of non-adjacent vertices of \vec{G}_*' must agree on at least one vertex and must disagree on at least one vertex. Note that any pair of non-adjacent vertices must be vertices of G and they agree on v_* . Thus, they must disagree on a vertex of G . Hence, the oriented graph induced by the vertices of G obtained from \vec{G}_*' is an oriented clique. Thus, G is an underlying oriented clique.

On the other hand, assume that G is an underlying oriented clique. Let \vec{G} be an orientation of G such that \vec{G} is an oriented clique. Now consider a orientation \vec{G}_* of G_* in which every vertex of G is an out-neighbor of v_* and the oriented graph induced by the vertices of G from \vec{G}_* is isomorphic to \vec{G} . It is easy to observe that \vec{G}_* is a push clique. \square

It is known that determining if a graph is an underlying oriented clique is NP-hard [1]. Hence, by the above lemma our result follows.

4 Proof of Theorem 1.2

The lower bound follows from the example of the planar push clique of order 8 depicted in Fig. 2(i).

Now we will prove the upper bound. We know that a push clique is an oriented graph with each pair of non-adjacent vertices agreeing on at least one other vertex and disagreeing on one other vertex. In particular, a push clique has diameter at most 2.

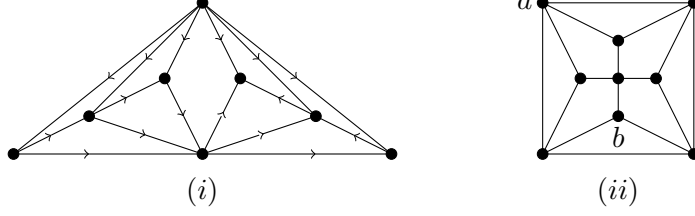


Figure 2: (i) A planar push clique of order 8, (ii) The unique diameter 2 planar graph with domination number 3 [3].

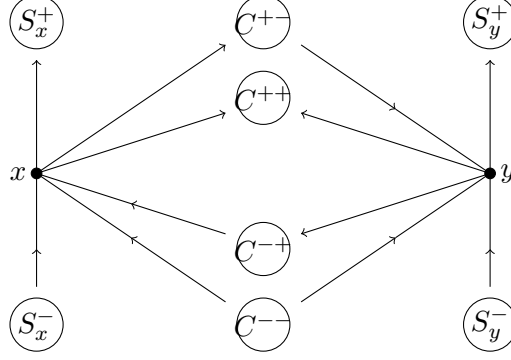


Figure 3: Structure of \vec{G} (not a planar embedding)

Goddard and Henning [3] showed that every planar graph with diameter 2 has domination number at most 2 except for a particular graph on nine vertices (see Fig. 2(ii)). It is easy check that vertices a and b in this graph are not connected by two distinct 2-paths. Therefore, it is not a push clique. Hence, a planar push cliques must have domination number at most 2.

Let \vec{B}' be a planar push clique dominated by the vertex v . Push all the in-neighbors of v to obtain the oriented graph \vec{B} from \vec{B}' . Note that \vec{B} is a push clique with $N_{\vec{B}}^+(v) = N_{\vec{B}}(v) = V(\vec{B}) \setminus \{v\}$. Observe that each pair of non-adjacent vertices from $N_{\vec{B}}(v)$ agrees on v and thus, must disagree on a vertex from $N_{\vec{B}}(v)$. Therefore, the graph induced by $N_{\vec{B}}(v)$ must be an oriented clique. Moreover, note that the oriented graph induced by $N_{\vec{B}}(v)$ is an outerplanar graph. We know that an outerplanar oriented clique can have at most seven vertices [10]. Thus, \vec{B} has order at most eight.

To prove Theorem 1.2 it will be enough to prove that any planar push clique with domination number 2 must have order at most 8. More precisely, we need to prove the following lemma.

Lemma 4.1. *Let \vec{H} be a planar push clique with domination number 2. Then $|V(\vec{H})| \leq 8$.*

Let \vec{G} be a planar push clique with $|V(\vec{G})| > 8$. Assume that \vec{G} is triangulated and has domination number 2.

We define the partial order \prec for the set of all dominating sets of order 2 of \vec{G} as follows: for any two dominating sets $D = \{x, y\}$ and $D' = \{x', y'\}$ of order 2 of \vec{G} , $D' \prec D$ if and only if $|N_{\vec{G}}(x') \cap N_{\vec{G}}(y')| < |N_{\vec{G}}(x) \cap N_{\vec{G}}(y)|$.

Let $D = \{x, y\}$ be a maximal dominating set of order 2 of \vec{G} with respect to \prec . Also for the rest of this proof, $t, t', \alpha, \bar{\alpha}, \beta, \bar{\beta}$ are variables satisfying $\{t, t'\} = \{x, y\}$ and $\{\alpha, \bar{\alpha}\} = \{\beta, \bar{\beta}\} = \{+, -\}$.

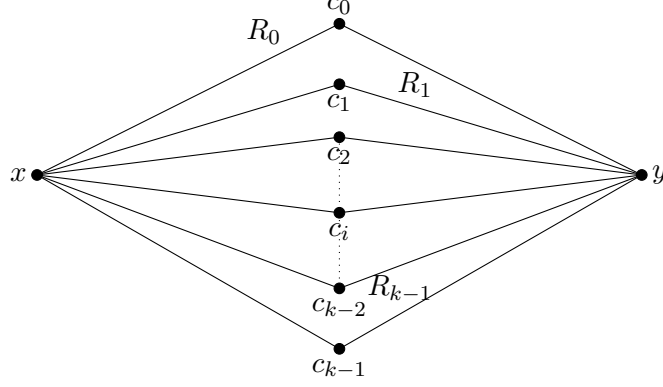


Figure 4: A planar embedding of $und(\vec{H})$

Now, we fix the following notations (Fig: 3):

$$C = N_{\vec{G}}(x) \cap N_{\vec{G}}(y), C^{\alpha\beta} = N_{\vec{G}}^{\alpha}(x) \cap N_{\vec{G}}^{\beta}(y), C_t^{\alpha} = N_{\vec{G}}^{\alpha}(t) \cap C,$$

$$S_t = N_{\vec{G}}(t) \setminus C, S_t^{\alpha} = S_t \cap N_{\vec{G}}^{\alpha}(t) \text{ and } S = S_x \cup S_y.$$

Without loss of generality, assume that, the oriented graph \vec{G} is such that $C_x = C_x^+$, $S_x = S_x^+$, $S_y = S_y^+$ and $|C_y^+| \geq |C_y^-|$. Note that, it is possible to obtain an alternative orientation of G from \vec{G} that satisfies the above conditions by pushing some vertices of \vec{G} . That is why our assumption is valid.

Hence we have,

$$9 \leq |\vec{G}| = |D| + |C| + |S|. \quad (1)$$

Let \vec{H} be the oriented graph obtained from the induced subgraph $\vec{G}[D \cup C]$ of \vec{G} by deleting all the arcs between the vertices of D and all the arcs between the vertices of C . Note that it is possible to extend the planar embedding of $und(\vec{H})$ given in Fig 4 to a planar embedding of $und(\vec{G})$ for some particular ordering of the elements of, say $C = \{c_0, c_1, \dots, c_{k-1}\}$.

Notice that $und(\vec{H})$ has k faces, namely the unbounded face F_0 and the faces F_i bounded by edges $xc_{i-1}, c_{i-1}y, yc_i, c_ix$ for $i \in \{1, \dots, k-1\}$. Geometrically, $und(\vec{H})$ divides the plane into k connected components. The *region* R_i of \vec{G} is the i^{th} connected component (corresponding to the face F_i) of the plane. *Boundary points* of a region R_i are c_{i-1} and c_i for $i \in \{1, \dots, k-1\}$ and, c_0 and c_{k-1} for $i = 0$. Two regions are *adjacent* if they have at least one common boundary point (hence, a region is adjacent to itself).

Now for the different possible values of $|C|$, we want to show that $und(\vec{H})$ cannot be extended to a planar push clique of order at least 9. Note that for extending $und(\vec{H})$ to \vec{G} we can add new vertices only from S . Any vertex $v \in S$ will be inside one of the regions R_i . If there is at least one vertex of S in a region R_i , then R_i is *non-empty* and *empty* otherwise. In fact, when there is no chance of confusion, R_i might represent the set of vertices of S contained in the region R_i .

First we will ask the question that “How small $|C|$ can be?” and prove the following lower bound of $|C|$.

Lemma 4.2. $|C| \geq 3$.

Proof. We know that x and y are either connected by two distinct 2-paths or by an arc. So, if x and y are non-adjacent, then we have $|C| \geq 2$. If x and y are adjacent, then the triangulation of \vec{G} implies $|C| \geq 2$.

To complete the proof we need to show that $|C| \neq 2$. We will prove by contradiction. Therefore, assume that $|C| = 2$. To get a contradiction to our assumption, by equation 1, it will be enough to show $|S| \leq 4$.

Note that, if we have $S_x = \emptyset$, then triangulation will force either multiple edges c_0c_1 (one in R_0 and one in R_1) or the edge xy making x a dominating vertex. Both contradicts our assumption. Hence we do not have $S_t = \emptyset$ for any $t \in \{x, y\}$.

Note that if for some $i \neq j$ both $S_x \cap R_i$ and $S_y \cap R_j$ are non-empty, then the vertices of the two sets must be adjacent to both c_0 and c_1 . This will imply $|S_x \cap R_i| \leq 1$ and $|S_y \cap R_j| \leq 1$.

Thus, if all the four sets $S_t \cap R_i \neq \emptyset$ for all $(t, i) \in \{x, y\} \times \{0, 1\}$, then $|S| \leq 4$.

Assume that we have exactly three non-empty sets $S_x \cap R_0$, $S_x \cap R_1$ and $S_y \cap R_0$ among the four sets $S_t \cap R_i$ for all $(t, i) \in \{x, y\} \times \{0, 1\}$. By triangulation we have the edge c_0c_1 inside R_1 and there is at least one vertex in $|S_x \cap R_0|$ adjacent to c_0 . Now we have the dominating set $\{x, c_0\}$ with at least three common neighbors (c_1 , a vertex from $S_x \cap R_0$ and a vertex from $S_x \cap R_1$) contradicting the maximality of D .

Therefore, exactly two sets among the four sets $S_t \cap R_i$ for all $(t, i) \in \{x, y\} \times \{0, 1\}$ can be non-empty. As $S_x, S_y \neq \emptyset$ we must have $S_x \cap R_i$ and $S_y \cap R_j$ as non-empty sets for some $i, j \in \{0, 1\}$. If $i \neq j$, then $|S| \leq 2$. Thus, we can assume that exactly the sets $S_x \cap R_0$ and $S_y \cap R_0$ are the two non-empty sets among the four sets $S_t \cap R_i$ for all $(t, i) \in \{x, y\} \times \{0, 1\}$.

For this case, assume that $S_x = \{x_1, x_2, \dots, x_{n_x}\}$ and $S_y = \{y_1, y_2, \dots, y_{n_y}\}$. Without loss of generality also assume that we have the edges $c_0x_1, x_1x_2, \dots, x_{n_x-1}x_{n_x}, x_{n_x}c_1$ and the edges $c_0y_1, y_1y_2, \dots, y_{n_y-1}y_{n_y}, y_{n_y}c_1$ by triangulation. Furthermore, we can assume $n_x \geq n_y$ without loss of generality.

Assume $n_y = 1$. So, to have $|S| \geq 5$ we should have $n_x \geq 4$. Note that we cannot have the edge xy as otherwise $\{y_1, x\}$ will be a dominating set with at least three common neighbors $\{c_0, c_1, y\}$ contradicting the maximality of D . Hence, we have the edge c_0c_1 inside R_1 by triangulation. Note that the vertex $x_2 \in S_x$ must be adjacent to either c_0 or c_1 or y_1 to have two distinct 2-paths connecting it to y . This will create a dominating set $\{c_0, x\}$ or $\{c_1, x\}$ or $\{y_1, x\}$ with at least three common neighbors $\{x_1, x_2, c_1\}$ or $\{x_{n_x}, x_2, c_0\}$ or $\{x_2, c_0, c_1\}$ respectively. This will contradict the maximality of D . Therefore, $n_y \geq 2$.

Now assume that we have the edge x_2c_0 . Then to have two distinct 2-paths connecting a vertex $w \in S_y$ to x_1 we will have y adjacent to both x_2 and c_0 . That means, each vertex of S_y will be adjacent to both x_2 and c_0 . But this is not possible keeping the graph planar as $n_y \geq 2$. So, there is no edge between c_0 and x_2 . By similar arguments, we can show that every t_i is non-adjacent to c_0 for $i \in \{2, 3, \dots, n_t\}$ and every t_j is non-adjacent to c_1 for $i \in \{1, 2, \dots, n_t - 1\}$ for all $t \in \{x, y\}$. With a similar argument we can also show that the edge t_it_{i+k} for $1 \leq i < i+k \leq n_t$ does not exist unless $k = 1$ for any $t \in \{x, y\}$.

Now notice that $n_x \geq 3$ by equation 1 and the assumption that $n_x \geq n_y$. By triangulation we must have the edge x_2y_i for some $i \in \{1, 2, \dots, n_y\}$. Then to have two distinct 2-paths between x_1 and y_j for $j \in \{i+1, \dots, n_y\}$ and to have two distinct 2-paths between x_3 and y_l for $l \in \{1, \dots, i-1\}$ we must have every vertex of S_y adjacent to x_2 .

If $n_y \geq 3$, then we cannot have two distinct 2-paths between the non-adjacent vertices x_1 and y_3 . So we must have $n_y = 2$.

Now to have two distinct 2-paths between the non-adjacent vertices x_1 and y_2 we must have

the edge x_1y_1 . This creates the dominating set $\{x, y_1\}$ with at least three common neighbors $\{c_0, x_1, x_2\}$ contradicting the maximality of D . Therefore, it is not possible to have $|C| = 2$. \square

Now we will ask the question that “How big $|C|$ can be?” and prove the following upper bound of $|C|$.

Lemma 4.3. $|C| \leq 3$.

Proof. First assume that $S = \emptyset$. Then $|C| = k \geq 7$ and there are at least four vertices of C that agree with each other on both x and y . Among those four vertices, two must be such that they are non-adjacent and the only common neighbors they have are x and y . Thus, those two non-adjacent vertices do not disagree on any vertex, a contradiction.

Now assume that $|C| = k \geq 4$ and $S \neq \emptyset$. Then, without loss of generality, assume a vertex $v \in S_x \cap R_0$. To have two distinct 2-paths connecting v to c_1 and c_{k-2} we must have the edges vc_0 and vc_{k-1} . Hence we have $|S_x \cap R_0| \leq 1$. In fact with a similar argument we can show that

$$|S_t \cap R_i| \leq 1 \text{ for all } (t, i) \in \{x, y\} \times \{0, 1, \dots, k-1\}. \quad (2)$$

Also note that if we have a vertex $v \in S_x \cap R_0$, then it is not possible to have any vertex in $S_y \cap R_i$ for all $i \in \{1, 2, \dots, k-1\}$ and in $S_x \cap R_j$ for all $j \in \{1, 2, \dots, k-1\}$. So basically, only adjacent regions can be non-empty.

Hence, at most two of the sets $S_t \cap R_i$ for all $(t, i) \in \{x, y\} \times \{0, 1, \dots, k-1\}$ can be non-empty. Then equation 2 implies $|S| \leq 2$. Then by equation 1 we have

$$9 \leq |G| \leq 2 + 4 + 2 = 8.$$

This is a contradiction. \square

Therefore, the only possible value for $|C|$ is 3. To prove Theorem 1.2 we will show that $|C| = 3$ is not possible in the following lemma.

Lemma 4.4. $|C| \neq 3$.

Proof. We will prove this lemma by contradiction. So, assume that $|C| = 3$. Also, without loss of generality, assume that $|S_x| \geq |S_y|$.

Note that by equation 1 we have $|S| \geq 5$. Hence we have $|S_x| \geq 3$.

First assume that $S_y = \emptyset$. Hence we do not have the edge xy as otherwise x will dominate the whole graph. Now note that any two regions are adjacent for $|C| = 3$. The vertices from different regions must be adjacent to their unique common boundary point to have two distinct 2-paths connecting them.

Hence, if we have all the regions non-empty, then we will have the vertices of each region adjacent to both the boundary points of that region. This will imply

$$|S_x \cap R_i| \leq 1 \text{ for all } i \in \{0, 1, 2\}.$$

This will imply $|S| \leq 3$ and contradict our assumption. Hence, it is not possible to have all the three regions non-empty when $S_y = \emptyset$.

If we have exactly two regions, say R_0 and R_1 , non-empty, then every vertex of S_x must be adjacent to c_0 to create two distinct 2-paths between the vertices of $S_x \cap R_0$ and the vertices

of $S_x \cap R_1$. This will create a dominating set $\{c_0, x\}$ with at least four common neighbors contradicting the maximality of D . Hence, we can have at most one region non-empty when $S_y = \emptyset$.

Now assume that exactly one region, say R_1 , is non-empty. Then each vertex of S_x must be adjacent to either c_0 or c_1 to have two distinct 2-paths connecting it to c_2 . Then, without loss of generality, we will have at least three vertices of S_x adjacent to c_0 . This will create a dominating set $\{c_0, x\}$ with at least four common neighbors (three vertices from S_x and c_2 because of triangulation) contradicting the maximality of D .

Hence $S_y \neq \emptyset$.

Now assume, without loss of generality, that $S_x \cap R_0 \neq \emptyset$. This implies $S_y \cap R_1 = \emptyset$ and $S_y \cap R_2 = \emptyset$ as it is not possible to have two distinct 2-paths between the vertices of S_x from one region and the vertices of S_y from a different region. But we also know that $S_y \neq \emptyset$. Hence we must have $S_y \cap R_0 \neq \emptyset$.

Now assume that $S_x = \{x_1, x_2, \dots, x_{n_x}\}$ and $S_y = \{y_1, y_2, \dots, y_{n_y}\}$. Without loss of generality also assume that we have the edges $c_0x_1, x_1x_2, \dots, x_{n_x-1}x_{n_x}, x_{n_x}c_1$ and the edges $c_0y_1, y_1y_2, \dots, y_{n_y-1}y_{n_y}, y_{n_y}c_1$ by triangulation.

Now x_2 must be adjacent to either c_0 or c_2 for having two distinct 2-paths connecting x_2 and c_1 . Without loss of generality assume that x_2 is adjacent to c_0 . Now each vertex of S_y must be adjacent to both x_2 and c_0 to have two distinct 2-paths connecting it to x_1 . But this contradicts the planarity of \vec{G} unless we have $n_y = 1$. Therefore, $n_y = 1$.

If $n_y = 1$, then $n_x \geq 4$ by equation 1. If we have the edge xy (say, inside region R_2), then each vertex of S_x must be adjacent to c_0 to have two distinct 2-paths connecting it to c_1 creating the dominating set $\{c_0, x\}$ with at least four common neighbors (the vertices of S_x) contradicting the maximality of D . Hence we do not have the edge xy .

Therefore, by triangulation, we must have the edges c_0c_1 and c_1c_2 . Now it is not possible to have more than two vertices of S_x adjacent to c_0 as it will create the dominating set $\{c_0, x\}$ contradicting the maximality of D . Similarly, it is not possible to have more than two vertices of S_x adjacent to c_2 . But as $|S_x| \geq 4$, by triangulation we must have (to avoid having more than two vertices from S_x adjacent to c_0 or c_2) at least two vertices of S_x adjacent to y_1 . This will create the dominating set $\{y_1, x\}$ with at least four common neighbors (c_0, c_2 and two vertices of S_x) contradicting the maximality of D .

Hence it is not possible to have $|C| = 3$. \square

The above three lemmas prove that for no value of $|C|$ it is possible to have a planar push clique with domination number 2 and order at least 9.

5 Proof of Theorem 1.3

We want to show that any planar underlying push clique must contain one of the graphs listed in Fig. 1 as a spanning subgraph. It is easy to observe that each graph listed in Fig. 1 is planar (we provided a planar drawing) and a push clique (it is easy to check). Note that by adding edges to a push clique one obtains another push clique. So to prove our result it is enough to show that if \vec{G} is a planar push clique with k vertices then its underlying graph G must contain one of the graphs with k vertices listed in Fig. 1 as its subgraph. From now on in this section whenever we mention a graph H_i for any $i \in \{1, 2, \dots, 18\}$, we mean the i th graph depicted in Fig. 1.

Let \vec{G} be a planar push clique with underlying graph G . Note that, G must be connected and any two non-adjacent vertices of G must have at least two common neighbors. Thus,

Observation 5.1. *Any underlying push clique that is not a complete graph must contain a 4-cycle as subgraph.*

If $|G| \leq 4$, then by Observation 5.1 we can say that G contains one of the four graphs H_1, H_2, H_3, H_4 as its spanning subgraph.

If $|G| = 5$, then G must contain a 4-cycle. The fifth vertex of G must be adjacent to at least two vertices of the 4-cycle. Thus, G either contains a $K_{2,3}$ or must contain a 5-cycle.

It is easy to observe that $K_{2,3}$ is not an underlying push clique. If we replace the partite set containing two vertices with an edge, then also the graph obtained is not an underlying push clique. But if we add an edge in the other partite set, a 5-cycle is created. Thus, G must contain a 5-cycle $abcdea$ (say).

Note that the 5-cycle $abcdea$ with two incident chords ac and ce is not an underlying push clique as b and d are neither adjacent nor they have at least two common neighbors. A 5-cycle and a 5-cycle with a single chord is a subgraph of the above mentioned graph, thus not underlying push cliques. Therefore, to obtain an underlying push clique from a 5-cycle we need at least two non-incident edges. The graph we get is H_5 .

Lemma 5.2. *If H is an underlying planar push clique of order $n \geq 6$ and has minimum degree 2, then H is isomorphic to H_6 .*

Proof. Let H be an underlying planar push clique of order $n \geq 6$. It is easy to note that any vertex of H must have degree at least two. If some vertex v of H has degree equal to two then each of its non-neighbor must be adjacent to both the neighbors of v , resulting in a $K_{2,n-2}$. Given any orientation of H we can push the neighbors of v , say x and y , in such a way that we have the arcs \vec{xv} and \vec{vy} . Thus, for being a push clique, each non-neighbor of v must be in a special 4-cycle with v while the other vertices of the cycle are x and y . Therefore, it is possible to push the non-neighbors of v to obtain an orientation of H such that \vec{xw} and \vec{yw} are arcs for each non-neighbor w of v . Let this so obtained orientation of H be \vec{H} . Now note that the only way for \vec{H} to be a planar push clique is to have the non-neighbors of v induce a 2-dipath. In that case, $n - 2 = 4$ and the underlying graph of \vec{H} is isomorphic to the graph H_6 . \square

If $|G| = 6$, then either G has minimum degree at least 3 or G is isomorphic to H_6 . In any case, G has a Hamiltonian cycle (Dirac's Theorem (1952) [11]), say, $abcdefa$.

Note that a six cycle can have two types of chords, namely, a *long chord* connecting vertices at distance 3 and a *short chord* connecting vertices at distance 2. We will obtain a minimal planar underlying push clique on 6 vertices by adding chords to the six cycle $abcdefa$ by case analysis.

If we do not add any long chord then we need to at least four short chords. The graph we obtain without adding any long chord is isomorphic to H_9 . If we add exactly one long chord then we obtain a graph isomorphic to H_7 or H_8 . If we add exactly two long chords then we obtain a graph isomorphic to H_6 . In fact, we do obtain some other graphs which contains one of the graphs H_7, H_8, H_9 as proper subgraphs and hence does not make it to our list. It is not possible to add three long chords keeping the graph planar.

Observation 5.3. *Each edge of the graphs H_6, H_7, \dots, H_{14} is part of a Hamiltonian cycle.*

If $|G| = 7$, then each vertex of G must have degree at least three by Lemma 5.2. If G has minimum degree 4 then G is Hamiltonian by Dirac's Theorem (1952) [11]. Otherwise, let v be a degree three vertex of G . Delete the vertex v from G and add the edges among its neighbors to obtain the graph G' . Note that as G was an underlying planar push clique on 7 vertices, G' must be an underlying planar push clique on 6 vertices. Thus, by what we have proved by now, G' must contain one of H_6, H_7, H_8, H_9 as its spanning subgraph. If that spanning underlying push clique of G' contains one of the edges among the neighbors of v then G is Hamiltonian by Observation 5.3. As the graphs H_7, H_8, H_9 have independence number 2, we will be done if G' has one of these graphs as its spanning subgraph. The graph H_6 has independence number 3 and has exactly one independent set of cardinality 3. If we add a vertex to the graph and make it adjacent to those three vertices, then a $K_{3,3}$ is created and thus the so obtained graph is not planar. Therefore, we can conclude that G is Hamiltonian. After that a routine case analysis assuming number of long chords will settle this case.

If $|G| = 8$, then each vertex of G must have degree at least three by Lemma 5.2. If G has minimum degree 4 then G is Hamiltonian by Dirac's Theorem (1952) [11]. The graphs H_{11}, H_{12}, H_{13} have independence number 2. The graphs H_{10}, H_{14} have independence number 3. Each of them has a unique independent set of cardinality 3. If we add a vertex to H_{10} or H_{14} and make it adjacent to the vertices of its unique independent set of cardinality 3, then a $K_{3,3}$ minor is created and thus the so obtained graph is not planar. Therefore, if G has minimum degree 3, then using Observation 5.3 and arguing exactly like the case above we can conclude that G is Hamiltonian.

Note that an eight cycle can have three types of chords, namely, a *very long chord* connecting vertices at distance 4, a long chord and a short chord. A routine case analysis assuming number of very long chords with subcases assuming the number of long chords will settle this case.

6 Conclusions

We listed all minimal planar underlying push cliques upto spanning subgraph inclusion. One can notice that there are four distinct minimal planar underlying push cliques of maximum order (eight vertices). This result is unlike the case of planar underlying oriented clique where there is a unique planar underlying oriented clique of maximum order [8].

The planar oriented cliques were instrumental in improving the bound of oriented chromatic number of planar graphs. Thus we hope that our list will help studies related to pushable chromatic number of oriented planar graphs.

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